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# "HCR's Infinite-series" Analysis of Oblique Frustum of a Right Circular Cone

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***Abstract:*** **Three series had been derived by the author, using double-integration in polar co-ordinates, binomial expansion and β & γ-functions for determining the volume, surface-area & perimeter of elliptical-section of oblique frustum of a right circular cone as there had not been any mathematical formula for determining the same due to some limitations. All these three series are in form of discrete summation of infinite terms which converge into finite values hence these were also named as HCR's convergence series. These are extremely useful in case studies & practical computations.**

***Keywords:*** **HCR's Convergence-series, oblique frustum, binomial expansion, β & γ-functions.**

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## I. INTRODUCTION

When a right circular cone is thoroughly cut by a plane inclined at a certain angle with the axis of a right circular cone, an oblique frustum with elliptical section & apex point is generated So far there are no mathematical formulae for determining the volume & surface area of oblique frustum & perimeter, major axis, minor axis & eccentricity of elliptical section. This article derives mathematical formulae for all the parameters.

## II. VOLUME OF OBLIQUE FRUSTUM

Let there be a right circular cone with apex angle $2\alpha$. Now, it is thoroughly cut by a plane (as shown by the extended line AB in fig 1.) inclined at an angle $\theta$ with axis OZ of the cone & lying at a normal distance h from the apex point O

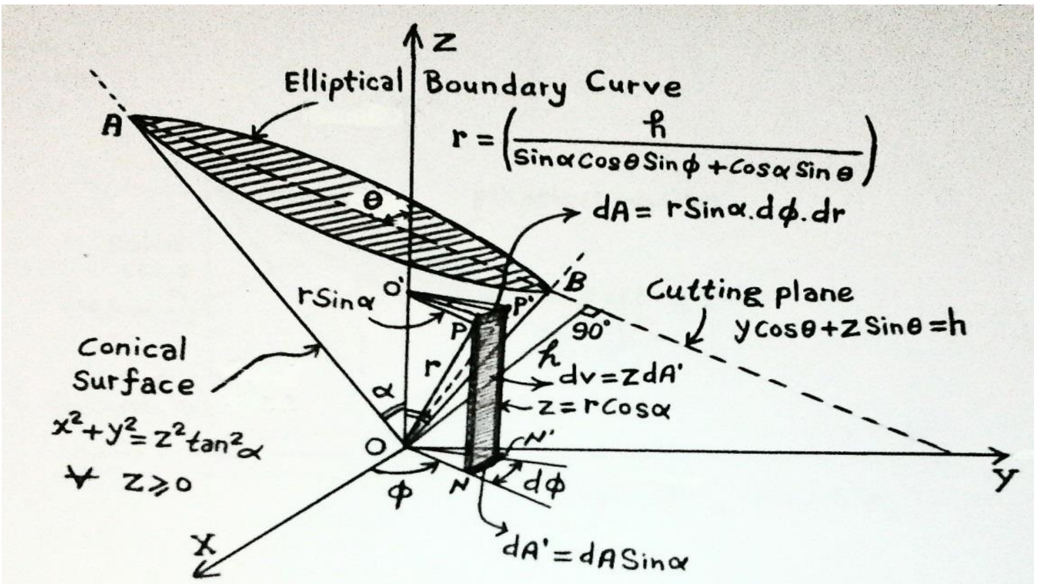


**Fig 1: Oblique frustum of right cone with elliptical section**

Now, consider any parametric point $P(r,\varphi)$ on the conical surface

**Equation of conical surface**: in Cartesian co-ordinates

$$x^2+y^2=(ztan\alpha)^2=z^2tan^2\alpha \quad \forall\, z\geq 0$$

In Polar co-ordinates

$$x=(rsin\alpha)cos\varphi$$

$$y=(rsin\alpha)sin\varphi$$

$$z=rcos\alpha$$

**Equation of cutting plane**: by intercept form

$$\frac{x}{\infty}+\frac{y}{hsec\theta}+\frac{z}{hcosec\theta}=1 \Rightarrow ycos\theta+zsin\theta=h$$

**Equation of elliptical boundary**: Ellipse is the curve of intersection of conical surface & cutting plane. Hence on substituting the co-ordinates of parametric point $P(r,\varphi)$ in the equation of plane, we have the following equation in polar co-ordinates

$$\{(rsin\alpha)sin\varphi\}cos\theta+\{rcos\alpha\}sin\theta=h$$

$$\Rightarrow \quad r=\frac{h}{(sin\alpha cos\theta)sin\varphi+sin\theta cos\alpha} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (I)$$

Now, let's consider an imaginary cylindrical surface ABDC completely enclosing the frustum of solid cone (as shown in the fig 2. below). It is easier to calculate the volume of void space enclosed by the frustum of cone, cylindrical surface & XY-plane.

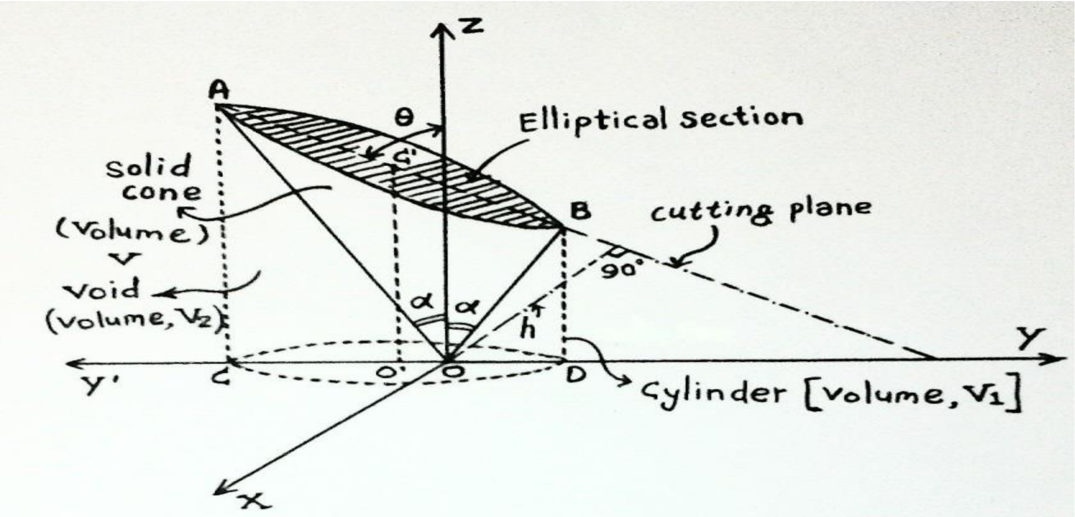


**Fig 2: Imaginary cylindrical surface enclosing the frustum of cone**

Let's consider an elementary surface area $dA$ on the conical surface (see fig 1.)

$$dA=\{(rsin\alpha)d\varphi\}dr=rsin\alpha d\varphi dr$$

Taking the projection $dA'$ of elementary area $dA$ on XY-plane

$$\Rightarrow dA^{'}=dAsin\alpha=(rsin\alpha d\varphi dr)sin\alpha=rsin^2\alpha d\varphi dr$$

Now, volume of elementary vertical strip

$$dV=dA^{'}\times z=dA^{'}\times rcos\alpha$$

$$\Rightarrow dV=(rsin^2\alpha d\varphi dr)rcos\alpha=r^2sin^2\alpha cos\alpha d\varphi dr$$

Hence, using double integration with proper limits, total volume ($V_2$) of the enclosed void space

$$\Rightarrow V_2 = \int dV = \int_{\varphi=0}^{\varphi=2\pi}\int_{r=0}^{r=r} r^2 sin^2\alpha cos\alpha d\varphi dr$$

$$= sin^2\alpha cos\alpha \int_{\varphi=0}^{\varphi=2\pi}\left[\int_{r=0}^{r=r} r^2 dr\right] d\varphi \qquad (since, \alpha = constant)$$

$$= sin^2\alpha cos\alpha \int_{\varphi=0}^{\varphi=2\pi}\left[\frac{r^3}{3}\right]_{r=0}^{r=r} d\varphi = \frac{sin^2\alpha cos\alpha}{3}\int_{\varphi=0}^{\varphi=2\pi} r^3 d\varphi$$

$On\ substituting\ the\ value\ of\ r\ from\ eq(I)$, we have

$$V_2 = \frac{sin^2\alpha cos\alpha}{3}\int_{\varphi=0}^{\varphi=2\pi}\left(\frac{h}{(sin\alpha cos\theta)sin\varphi + sin\theta cos\alpha}\right)^3 d\varphi$$

$$= \frac{sin^2\alpha cos\alpha}{3}\int_{\varphi=0}^{\varphi=2\pi}\frac{h^3}{(sin\theta cos\alpha)^3\left(1+\left(\frac{sin\alpha cos\theta}{sin\theta cos\alpha}\right)sin\varphi\right)^3} d\varphi$$

$$= \frac{h^3 sin^2\alpha cos\alpha}{3(sin\theta cos\alpha)^3}\int_{\varphi=0}^{\varphi=2\pi}\frac{1}{\left(1+\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right)^3} d\varphi \qquad (h, \alpha\ \&\ \theta\ are\ arbitrary\ constants)$$

$$= \frac{h^3 tan^2\alpha}{3sin^3\theta}\int_{\varphi=0}^{\varphi=2\pi}\left(1+\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right)^{-3} d\varphi$$

Since, the frustum has **finite elliptical section**, hence we have a condition

$$0 < \alpha < \theta \le \frac{\pi}{2} \ \Rightarrow \frac{tan\alpha}{tan\theta} < 1$$

$$\therefore \left|\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right| < 1 \qquad \forall\ 0 \le \varphi \le 2\pi$$

Now, using **Binomial Expansion**

$$\boldsymbol{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\dots\dots\dots\dots.\infty} \qquad (\forall\ |x| < 1)$$

We have,

$$\Rightarrow V_2 = \frac{h^3 tan^2\alpha}{3sin^3\theta}\int_{\varphi=0}^{\varphi=2\pi}(1+Csin\varphi)^{-3}d\varphi \qquad \left(let\ \frac{tan\alpha}{tan\theta} = constant = C\right)$$

$$= \frac{h^3 tan^2\alpha}{3sin^3\theta}\int_{\varphi=0}^{\varphi=2\pi}\Big[1 - 3(Csin\varphi) + \frac{3\times4}{2!}(Csin\varphi)^2 - \frac{3\times4\times5}{3!}(Csin\varphi)^3 + \frac{3\times4\times5\times6}{4!}(Csin\varphi)^4$$
$$- \frac{3\times4\times5\times6\times7}{5!}(Csin\varphi)^5 + \dots\dots\dots\dots\dots\dots\ \infty\Big] d\varphi$$

$$= \frac{h^3 tan^2\alpha}{3sin^3\theta}\Big[\int_{\varphi=0}^{\varphi=2\pi}1.d\varphi - \frac{1\times2\times3}{2}C\int_{\varphi=0}^{\varphi=2\pi}sin\varphi d\varphi + \frac{1\times2\times3\times4}{2\times2!}C^2\int_{\varphi=0}^{\varphi=2\pi}sin^2\varphi d\varphi$$
$$- \frac{1\times2\times3\times4\times5}{2\times3!}C^3\int_{\varphi=0}^{\varphi=2\pi}sin^3\varphi d\varphi + \frac{1\times2\times3\times4\times5\times6}{2\times4!}C^4\int_{\varphi=0}^{\varphi=2\pi}sin^4\varphi d\varphi$$
$$- \frac{1\times2\times3\times4\times5\times6\times7}{2\times5!}C^5\int_{\varphi=0}^{\varphi=2\pi}sin^5\varphi d\varphi + \dots\dots\dots\dots\ \infty\Big]$$

(Multiplying & diving each term by 2 in above series)

$$= \frac{h^3 tan^2\alpha}{3sin^3\theta}\left[2\pi - \frac{3!}{2}C\int_{\varphi=0}^{\varphi=2\pi} sin\varphi d\varphi + \frac{4!}{2\times 2!}C^2\int_{\varphi=0}^{\varphi=2\pi} sin^2\varphi d\varphi - \frac{5!}{2\times 3!}C^3\int_{\varphi=0}^{\varphi=2\pi} sin^3\varphi d\varphi + \frac{6!}{2\times 4!}C^4\int_{\varphi=0}^{\varphi=2\pi} sin^4\varphi d\varphi - \frac{7!}{2\times 5!}C^5\int_{\varphi=0}^{\varphi=2\pi} sin^5\varphi d\varphi + \dots\dots\dots\dots \infty\right]$$

$$\boldsymbol{since}, \int_0^{2a} \boldsymbol{f(x)\,dx = 0 \; if \; f(2a - x) = -f(x)}$$

$$\boldsymbol{= 2\int_0^{a} f(x)\,dx \;\; if \; f(2a - x) = f(x)}$$

$$\therefore \int_{\varphi=0}^{\varphi=2\pi} sin\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi} sin^3\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi} sin^5\varphi d\varphi = \dots\dots\dots.. = 0 \;\; \&$$

$$\int_0^{2\pi} sin^{2n}\varphi d\varphi = 2\int_0^{\pi} sin^{2n}\varphi d\varphi = 4\int_0^{\frac{\pi}{2}} sin^{2n}\varphi d\varphi \qquad \forall\, n \in N$$

Now, we have

$$V_2 = \frac{h^3 tan^2\alpha}{3sin^3\theta}\left[2\pi + 4\frac{4!}{2\times 2!}C^2\int_0^{\frac{\pi}{2}} sin^2\varphi d\varphi + 4\frac{6!}{2\times 4!}C^4\int_0^{\frac{\pi}{2}} sin^4\varphi d\varphi + 4\frac{8!}{2\times 6!}C^6\int_0^{\frac{\pi}{2}} sin^6\varphi d\varphi + \dots\dots\dots\dots \infty\right]$$

$$= \frac{h^3 tan^2\alpha}{3sin^3\theta}\left[2\pi + \frac{2\times 4!}{2!}C^2\int_0^{\frac{\pi}{2}} sin^2\varphi d\varphi + \frac{2\times 6!}{4!}C^4\int_0^{\frac{\pi}{2}} sin^4\varphi d\varphi + \frac{2\times 8!}{6!}C^6\int_0^{\frac{\pi}{2}} sin^6\varphi d\varphi + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2h^3 tan^2\alpha}{3sin^3\theta}\left[\pi + \frac{4!}{2!}C^2\int_0^{\frac{\pi}{2}} sin^2\varphi d\varphi + \frac{6!}{4!}C^4\int_0^{\frac{\pi}{2}} sin^4\varphi d\varphi + \frac{8!}{6!}C^6\int_0^{\frac{\pi}{2}} sin^6\varphi d\varphi + \dots\dots\dots\dots \infty\right]$$

Since, we know from $\boldsymbol{\beta}$ & $\gamma$**-functions**

$$\int_0^{\frac{\pi}{2}} \boldsymbol{sin^m x cos^n x\, dx} = \frac{\boldsymbol{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}}{\boldsymbol{2\Gamma\left(\frac{m+n+2}{2}\right)}} \;\; \& \; \Gamma\left(\frac{\boldsymbol{1}}{\boldsymbol{2}}\right) = \sqrt{\boldsymbol{\pi}}$$

Hence, we have

$$V_2 = \frac{2h^3 tan^2\alpha}{3sin^3\theta}\left[\pi + \frac{4!}{2!}\left(\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2+0+2}{2}\right)}\right)C^2 + \frac{6!}{4!}\left(\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)}\right)C^4 + \frac{8!}{6!}\left(\frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2h^3 tan^2\alpha}{3sin^3\theta}\left[\pi + \frac{4!}{2!}\left(\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)}\right)C^2 + \frac{6!}{4!}\left(\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)}\right)C^4 + \frac{8!}{6!}\left(\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2h^3 tan^2\alpha}{3sin^3\theta}\left[\pi + \frac{4!}{2!}\left(\frac{\frac{1}{2}\sqrt{\pi}\times\sqrt{\pi}}{2\times 1!}\right)C^2 + \frac{6!}{4!}\left(\frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\sqrt{\pi}}{2\times 2!}\right)C^4 + \frac{8!}{6!}\left(\frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\sqrt{\pi}}{2\times 3!}\right)C^6 + \dots\dots\dots\dots\infty\right]$$

$$= \frac{2h^3 tan^2\alpha}{3sin^3\theta}\left[\pi + \frac{4!}{2!}\times\frac{\pi}{2}\left(\frac{\frac{1}{2}}{1!}\right)C^2 + \frac{6!}{4!}\times\frac{\pi}{2}\left(\frac{\frac{3}{2}\times\frac{1}{2}}{2!}\right)C^4 + \frac{8!}{6!}\times\frac{\pi}{2}\left(\frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}}{3!}\right)C^6 + \dots\dots\dots\dots\infty\right]$$

$$= \frac{2h^3 tan^2\alpha}{3sin^3\theta}\times\pi\left[1 + \frac{4!}{2\times 2!}\left(\frac{1}{2\times 1!}\right)C^2 + \frac{6!}{2\times 4!}\left(\frac{1\times 3}{2^2\times 2!}\right)C^4 + \frac{8!}{2\times 6!}\left(\frac{1\times 3\times 5}{2^3\times 3!}\right)C^6 + \dots\dots\dots\dots\infty\right]$$

$$= \frac{2\pi h^3 tan^2\alpha}{3sin^3\theta}\left[1 + \frac{4!}{2\times 2!}\left(\frac{1}{2\times 1!}\right)C^2 + \frac{6!}{2\times 4!}\left(\frac{1\times 3}{2^2\times 2!}\right)C^4 + \frac{8!}{2\times 6!}\left(\frac{1\times 3\times 5}{2^3\times 3!}\right)C^6 + \dots\dots\dots\dots\infty\right]$$

$$\Rightarrow V_2 = \left(\frac{2\pi h^3 tan^2\alpha}{3sin^3\theta}\right)F_V \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots (II)$$

Where, $F_V \rightarrow$ is called Factor of volume or **HCR's $F_V$ series**

$$F_V = 1 + \frac{4!}{2\times 2!}\left(\frac{1}{2\times 1!}\right)C^2 + \frac{6!}{2\times 4!}\left(\frac{1\times 3}{2^2\times 2!}\right)C^4 + \frac{8!}{2\times 6!}\left(\frac{1\times 3\times 5}{2^3\times 3!}\right)C^6 + \dots\dots\dots\dots\infty$$

$$= 1 + \frac{1}{2}\left[\frac{4!}{2!}\left(\frac{1}{2\times 1!}\right)C^2 + \frac{6!}{4!}\left(\frac{1\times 3}{2^2\times 2!}\right)C^4 + \frac{8!}{6!}\left(\frac{1\times 3\times 5}{2^3\times 3!}\right)C^6 + \frac{10!}{8!}\left(\frac{1\times 3\times 5\times 7}{2^4\times 4!}\right)C^8 + \dots\dots\dots\dots\infty\right]$$

**In generalised form**

$$F_V = 1 + \frac{1}{2}\sum_{n=1}^{\infty}\left[\frac{(2n+2)!}{(2n)!}\left\{\frac{1\times 3\times 5\times\dots\dots\dots.(2n-3)\times(2n-1)}{2^n\times n!}\right\}C^{2n}\right]$$

$$= 1 + \sum_{n=1}^{\infty}\left[\frac{(2n+2)\times(2n+1)\times(2n)!}{2\times(2n)!}\left\{\frac{1\times 3\times 5\times\dots\dots\dots.(2n-3)\times(2n-1)}{2^n\times n!}\right\}C^{2n}\right]$$

$$\Rightarrow \boldsymbol{F_V = 1 + \sum_{n=1}^{\infty}\left[(n+1)(2n+1)\left\{\frac{1\times 3\times 5\times\dots\dots\dots.(2n-3)\times(2n-1)}{2^n\times n!}\right\}C^{2n}\right]} \qquad \dots\dots\dots\dots.(III)$$

Now, the **volume $(V_1)$ of imaginary cylinder** enclosing the frustum of solid cone (see fig 2)

$$V_1 = \frac{1}{2}[\pi(O'D)^2(AC+BD)]$$

$$\Rightarrow O'D = BC'sin\theta$$

$$where, BC' = \boldsymbol{semi\ major\ axis\ of\ ellipse} = \frac{\boldsymbol{hsin\alpha cos\alpha}}{\boldsymbol{(sin^2\theta - sin^2\alpha)}}$$

$$BD = \frac{hcos\alpha}{sin(\alpha+\theta)} \quad \&$$

$$AC = BD + ABcos\theta = \frac{hcos\alpha}{sin(\alpha+\theta)} + \frac{2hsin\alpha cos\alpha cos\theta}{(sin^2\theta - sin^2\alpha)}$$

Now, on setting the values, we get the volume of cylinder

$$V_1 = \frac{1}{2}\left[\pi\left(\frac{hsin\alpha cos\alpha sin\theta}{(sin^2\theta - sin^2\alpha)}\right)^2\left(\frac{2hcos\alpha}{sin(\alpha+\theta)} + \frac{2hsin\alpha cos\alpha cos\theta}{(sin^2\theta - sin^2\alpha)}\right)\right]$$

$$= \frac{\pi h^3 sin^2\theta sin^2\alpha cos^2\alpha}{(sin^2\theta - sin^2\alpha)^2}\left(\frac{1}{sin(\alpha+\theta)} + \frac{sin\alpha cos\theta}{(sin^2\theta - sin^2\alpha)}\right)$$

$$\Rightarrow V_1 = \frac{\pi h^3 sin^3\theta sin^2\alpha cos^4\alpha}{(sin^2\theta - sin^2\alpha)^3}$$

Hence, the volume (V) of the **oblique frustum** of right cone

$$V = V_1 - V_2$$

$$\mathbf{V} = \frac{\pi h^3 sin^3\theta sin^2\alpha cos^4\alpha}{(sin^2\theta - sin^2\alpha)^3} - \left(\frac{2\pi h^3 tan^2\alpha}{3sin^3\theta}\right)F_V \qquad \ldots\ldots\ldots\ldots\ldots\ldots(IV)$$

Above is the required expression for calculating the volume of oblique frustum.

$F_V$ series can be simplified as follows

$$F_V = 1 + 3C^2 + 5.625C^4 + 8.75C^6 + 12.3046875C^8 + 16.2421875C^{10} + 20.52832031C^{12} + 25.13671875C^{14} \ldots\ldots\ldots\ldots\ldots\ldots.\infty$$

## III. SURFACE AREA OF OBLIQUE FRUSTUM

We know that the elementary area $dA$ conical surface (see fig 1.)

$$dA = \{(rsin\alpha)d\varphi\}dr = rsin\alpha d\varphi dr$$

Hence, total surface area (S) of frustum of cone (using double integration with proper limits)

$$S = \int dA = \int_{\varphi=0}^{\varphi=2\pi}\int_{r=0}^{r=r} rsin\alpha d\varphi dr$$

$$= sin\alpha\int_{\varphi=0}^{\varphi=2\pi}\left[\int_{r=0}^{r=r} rdr\right]d\varphi$$

$$= sin\alpha\int_{\varphi=0}^{\varphi=2\pi}\left[\frac{r^2}{2}\right]_{r=0}^{r=r}d\varphi = \frac{sin\alpha}{2}\int_{\varphi=0}^{\varphi=2\pi} r^2 d\varphi$$

$On\ substituting\ the\ value\ of\ r\ from\ eq(I)$, we have

$$S = \frac{sin\alpha}{2}\int_{\varphi=0}^{\varphi=2\pi}\left(\frac{h}{(sin\alpha cos\theta)sin\varphi + sin\theta cos\alpha}\right)^2 d\varphi$$

$$= \frac{sin\alpha}{2}\int_{\varphi=0}^{\varphi=2\pi}\frac{h^2}{(sin\theta cos\alpha)^2\left(1 + \left(\frac{sin\alpha cos\theta}{sin\theta cos\alpha}\right)sin\varphi\right)^2}d\varphi$$

$$= \frac{h^2 sin\alpha}{2(sin\theta cos\alpha)^2}\int_{\varphi=0}^{\varphi=2\pi}\frac{1}{\left(1 + \left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right)^2}d\varphi \qquad (h, \alpha\ \&\ \theta\ are\ arbitrary\ constants)$$

$$= \frac{h^2 sec\alpha tan\alpha}{2sin^2\theta}\int_{\varphi=0}^{\varphi=2\pi}\left(1+\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right)^{-2} d\varphi$$

$$since \quad \left|\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right| < 1 \qquad \forall\ 0 \le \varphi \le 2\pi$$

$$Hence, using\ \textbf{binomial expansion}\ of\ \left(1+\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right)^{-2}$$

We have,

$$\Rightarrow\ S = \frac{h^2 sec\alpha tan\alpha}{2sin^2\theta}\int_{\varphi=0}^{\varphi=2\pi}(1+Csin\varphi)^{-2}d\varphi \qquad \left(let\ \frac{tan\alpha}{tan\theta} = constant = C\right)$$

$$S = \frac{h^2 sec\alpha tan\alpha}{2sin^2\theta}\int_{\varphi=0}^{\varphi=2\pi}\Big[1-2(Csin\varphi)+\frac{2\times3}{2!}(Csin\varphi)^2-\frac{2\times3\times4}{3!}(Csin\varphi)^3+\frac{2\times3\times4\times5}{4!}(Csin\varphi)^4 - \frac{2\times3\times4\times5\times6}{5!}(Csin\varphi)^5 + \dots\dots\dots\dots\dots\dots\ \infty\Big]d\varphi$$

$$= \frac{h^2 sec\alpha tan\alpha}{2sin^2\theta}\Big[\int_{\varphi=0}^{\varphi=2\pi}1.d\varphi - 1\times2C\int_{\varphi=0}^{\varphi=2\pi}sin\varphi d\varphi + \frac{1\times2\times3}{2!}C^2\int_{\varphi=0}^{\varphi=2\pi}sin^2\varphi d\varphi - \frac{1\times2\times3\times4}{3!}C^3\int_{\varphi=0}^{\varphi=2\pi}sin^3\varphi d\varphi + \frac{1\times2\times3\times4\times5}{4!}C^4\int_{\varphi=0}^{\varphi=2\pi}sin^4\varphi d\varphi - \frac{1\times2\times3\times4\times5\times6}{5!}C^5\int_{\varphi=0}^{\varphi=2\pi}sin^5\varphi d\varphi + \dots\dots\dots\dots\ \infty\Big]$$

$$= \frac{h^2 sec\alpha tan\alpha}{2sin^2\theta}\Big[2\pi - 2!\,C\int_{\varphi=0}^{\varphi=2\pi}sin\varphi d\varphi + \frac{3!}{2!}C^2\int_{\varphi=0}^{\varphi=2\pi}sin^2\varphi d\varphi - \frac{4!}{3!}C^3\int_{\varphi=0}^{\varphi=2\pi}sin^3\varphi d\varphi + \frac{5!}{4!}C^4\int_{\varphi=0}^{\varphi=2\pi}sin^4\varphi d\varphi - \frac{6!}{5!}C^5\int_{\varphi=0}^{\varphi=2\pi}sin^5\varphi d\varphi + \dots\dots\dots\dots\ \infty\Big]$$

$$\textbf{since}, \int_0^{2a} f(x)\,dx = 0\ if\ f(2a-x) = -f(x)$$

$$= 2\int_0^{a} f(x)\,dx\ \ if\ f(2a-x) = f(x)$$

$$\therefore \int_{\varphi=0}^{\varphi=2\pi}sin\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi}sin^3\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi}sin^5\varphi d\varphi = \dots\dots\dots.. = 0 \ \ \&$$

$$\int_0^{2\pi}sin^{2n}\varphi d\varphi = 2\int_0^{\pi}sin^{2n}\varphi d\varphi = 4\int_0^{\frac{\pi}{2}}sin^{2n}\varphi d\varphi \qquad \forall\ n \in N$$

Now, we have

$$S = \frac{h^2 sec\alpha tan\alpha}{2sin^2\theta}\left[2\pi + \frac{4\times3!}{2!}C^2\int_0^{\frac{\pi}{2}}sin^2\varphi d\varphi + \frac{4\times5!}{4!}C^4\int_0^{\frac{\pi}{2}}sin^4\varphi d\varphi + \frac{4\times7!}{6!}C^6\int_0^{\frac{\pi}{2}}sin^6\varphi d\varphi \quad + \dots\dots\dots\dots\ \infty\right]$$

$$= \frac{h^2 secatan\alpha}{sin^2\theta}\left[\pi + \frac{2 \times 3!}{2!}C^2\int_0^{\frac{\pi}{2}} sin^2\varphi d\varphi + \frac{2 \times 5!}{4!}C^4\int_0^{\frac{\pi}{2}} sin^4\varphi d\varphi + \frac{2 \times 7!}{6!}C^6\int_0^{\frac{\pi}{2}} sin^6\varphi d\varphi \quad + \dots\dots\dots\dots \infty\right]$$

Using $\boldsymbol{\beta}$ & $\gamma$**-functions**

$$\int_0^{\frac{\pi}{2}} \boldsymbol{sin^m x cos^n x\, dx} = \frac{\boldsymbol{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}}{\boldsymbol{2\Gamma\left(\frac{m+n+2}{2}\right)}} \quad \& \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence, we have

$$S = \frac{h^2 secatan\alpha}{sin^2\theta}\left[\pi + \frac{2 \times 3!}{2!}\left(\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2+0+2}{2}\right)}\right)C^2 + \frac{2 \times 5!}{4!}\left(\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)}\right)C^4 + \frac{2 \times 7!}{6!}\left(\frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{h^2 secatan\alpha}{sin^2\theta}\left[\pi + \frac{2 \times 3!}{2!}\left(\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)}\right)C^2 + \frac{2 \times 5!}{4!}\left(\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)}\right)C^4 + \frac{2 \times 7!}{6!}\left(\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{h^2 secatan\alpha}{sin^2\theta}\left[\pi + \frac{2 \times 3!}{2!}\left(\frac{\frac{1}{2}\sqrt{\pi} \times \sqrt{\pi}}{2 \times 1!}\right)C^2 + \frac{2 \times 5!}{4!}\left(\frac{\frac{3}{2} \times \frac{1}{2}\sqrt{\pi} \times \sqrt{\pi}}{2 \times 2!}\right)C^4 + \frac{2 \times 7!}{6!}\left(\frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\sqrt{\pi} \times \sqrt{\pi}}{2 \times 3!}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{h^2 secatan\alpha}{sin^2\theta}\left[\pi + \frac{2 \times 3!}{2!} \times \frac{\pi}{2}\left(\frac{\frac{1}{2}}{1!}\right)C^2 + \frac{2 \times 5!}{4!} \times \frac{\pi}{2}\left(\frac{\frac{3}{2} \times \frac{1}{2}}{2!}\right)C^4 + \frac{27 \times !}{6!} \times \frac{\pi}{2}\left(\frac{\frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}}{3!}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{h^2 secatan\alpha}{sin^2\theta} \times \pi\left[1 + \frac{3!}{2!}\left(\frac{1}{2 \times 1!}\right)C^2 + \frac{5!}{4!}\left(\frac{1 \times 3}{2^2 \times 2!}\right)C^4 + \frac{7!}{6!}\left(\frac{1 \times 3 \times 5}{2^3 \times 3!}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{\pi h^2 secatan\alpha}{sin^2\theta}\left[1 + \frac{3!}{2!}\left(\frac{1}{2 \times 1!}\right)C^2 + \frac{5!}{4!}\left(\frac{1 \times 3}{2^2 \times 2!}\right)C^4 + \frac{7!}{6!}\left(\frac{1 \times 3 \times 5}{2^3 \times 3!}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$\Rightarrow \boldsymbol{S} = \left(\frac{\boldsymbol{\pi h^2 secatan\alpha}}{\boldsymbol{sin^2\theta}}\right)\boldsymbol{F_S} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots.. \ (V)$$

Above is the required expression for calculating the surface area of oblique frustum.

Where, $F_S \rightarrow$ is called Factor of surface area or **HCR's $\boldsymbol{F_S}$ series**

$$F_S = 1 + \frac{3!}{2!}\left(\frac{1}{2 \times 1!}\right)C^2 + \frac{5!}{4!}\left(\frac{1 \times 3}{2^2 \times 2!}\right)C^4 + \frac{7!}{6!}\left(\frac{1 \times 3 \times 5}{2^3 \times 3!}\right)C^6 + \dots\dots\dots\dots \infty$$

**In generalised form**

$$F_S = 1 + \sum_{n=1}^{\infty}\left[\frac{(2n+1)!}{(2n)!}\left\{\frac{1 \times 3 \times 5 \times \dots\dots\dots.(2n-3) \times (2n-1)}{2^n \times n!}\right\}C^{2n}\right]$$

$$= 1 + \sum_{n=1}^{\infty}\left[\frac{(2n+1)\times(2n)!}{(2n)!}\left\{\frac{1\times3\times5\times\ldots\ldots\ldots.(2n-3)\times(2n-1)}{2^n\times n!}\right\}C^{2n}\right]$$

$$\Rightarrow \boldsymbol{F_S = 1 + \sum_{n=1}^{\infty}\left[(2n+1)\left\{\frac{1\times3\times5\times\ldots\ldots\ldots.(2n-3)\times(2n-1)}{2^n\times n!}\right\}C^{2n}\right]} \quad \ldots\ldots\ldots\ldots.(VI)$$

$F_S$ series can be simplified as follows

$$\boldsymbol{F_S = 1 + 1.5C^2 + 1.875C^4 + 2.1875C^6 + 2.4609375C^8 + 2.70703125C^{10} + 2.932617188C^{12} + 3.142089844C^{14} +} \ldots\ldots\ldots\ldots\ldots\ldots.\infty$$

## IV. PERIMETER OF ELLIPTICAL SECTION OF OBLIQUE FRUSTUM

Now, consider any parametric point $P(r, \varphi)$ on the periphery of elliptical section (see fig 3. below)

Now, the elementary perimeter $PP' = dP = O'P \times \angle PO'P'$

$$dP = (r sin\alpha) \times \left(\frac{d\varphi}{sin\theta}\right) = \frac{r sin\alpha d\varphi}{sin\theta}$$

Hence, the total perimeter (P) of the elliptical section (using integration with proper limits)

$$P = \int dP = \int_{\varphi=0}^{\varphi=2\pi} \frac{r sin\alpha d\varphi}{sin\theta}$$

$$= \frac{sin\alpha}{sin\theta}\int_{\varphi=0}^{\varphi=2\pi} r d\varphi$$

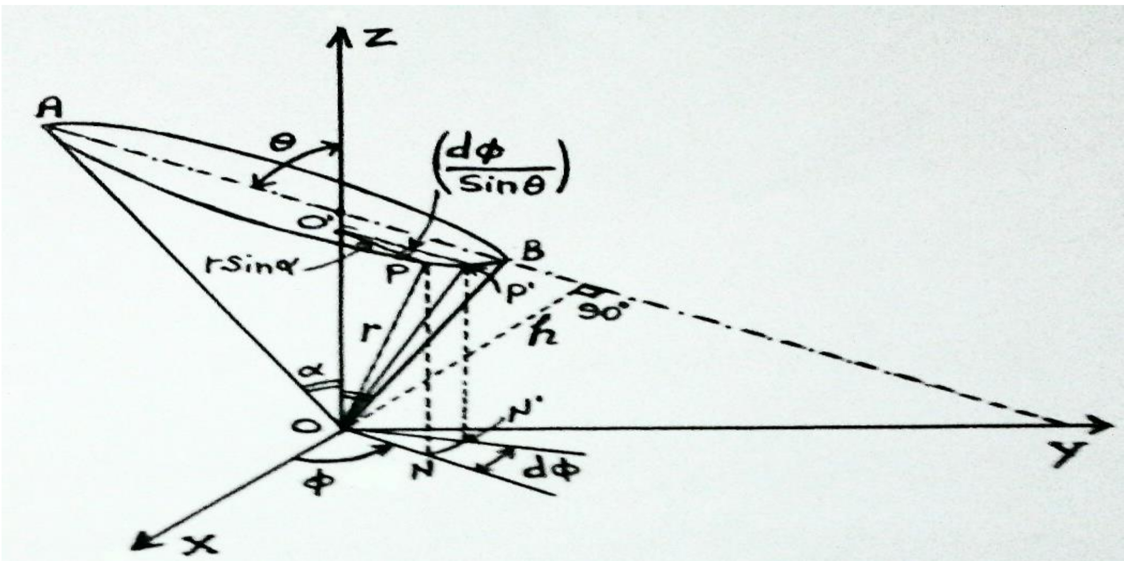


**Fig 3: Perimeter of elliptical section of oblique frustum**

*On substituting the value of r from eq(I)*, we have

$$P = \frac{sin\alpha}{sin\theta}\int_{\varphi=0}^{\varphi=2\pi}\left(\frac{h}{(sin\alpha cos\theta)sin\varphi + sin\theta cos\alpha}\right)d\varphi$$

$$= \frac{sin\alpha}{sin\theta}\int_{\varphi=0}^{\varphi=2\pi}\frac{h}{(sin\theta cos\alpha)\left(1+\left(\frac{sin\alpha cos\theta}{sin\theta cos\alpha}\right)sin\varphi\right)}d\varphi$$

$$= \frac{hsin\alpha}{sin\theta(sin\theta cos\alpha)}\int_{\varphi=0}^{\varphi=2\pi}\frac{1}{\left(1+\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right)}d\varphi \qquad (h, \alpha \;\&\; \theta \; are \; arbitrary \; constants)$$

$$= \frac{htan\alpha}{sin^2\theta}\int_{\varphi=0}^{\varphi=2\pi}\left(1+\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right)^{-1}d\varphi$$

$$since \;\; \left|\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right| < 1 \qquad \forall \;\; 0 \le \varphi \le 2\pi$$

$$Hence, using \; \textbf{binomial expansion} \; of \; \left(1+\left(\frac{tan\alpha}{tan\theta}\right)sin\varphi\right)^{-1} \; as \; follows$$

$$\mathbf{(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 -} \; \ldots\ldots\ldots\ldots\ldots\ldots\infty \qquad (\forall \;\; |\boldsymbol{x}| < 1)$$

We have,

$$\Rightarrow \; P = \frac{htan\alpha}{sin^2\theta}\int_{\varphi=0}^{\varphi=2\pi}(1+Csin\varphi)^{-1}d\varphi \qquad \left(let \;\; \frac{tan\alpha}{tan\theta} = constant = C\right)$$

$$P = \frac{htan\alpha}{sin^2\theta}\int_{\varphi=0}^{\varphi=2\pi}[1-(Csin\varphi)+(Csin\varphi)^2-(Csin\varphi)^3+(Csin\varphi)^4 \;\; -(Csin\varphi)^5 + \;\ldots\ldots\ldots\ldots\ldots\ldots\;\infty]d\varphi$$

$$= \frac{htan\alpha}{sin^2\theta}\left[\int_{\varphi=0}^{\varphi=2\pi}1.d\varphi - C\int_{\varphi=0}^{\varphi=2\pi}sin\varphi d\varphi + C^2\int_{\varphi=0}^{\varphi=2\pi}sin^2\varphi d\varphi - C^3\int_{\varphi=0}^{\varphi=2\pi}sin^3\varphi d\varphi + C^4\int_{\varphi=0}^{\varphi=2\pi}sin^4\varphi d\varphi - C^5\int_{\varphi=0}^{\varphi=2\pi}sin^5\varphi d\varphi + C^6\int_{\varphi=0}^{\varphi=2\pi}sin^6\varphi d\varphi - \;\ldots\ldots\ldots\ldots\;\infty\right]$$

$$= \frac{htan\alpha}{sin^2\theta}\left[2\pi - C\int_{\varphi=0}^{\varphi=2\pi}sin\varphi d\varphi + C^2\int_{\varphi=0}^{\varphi=2\pi}sin^2\varphi d\varphi - C^3\int_{\varphi=0}^{\varphi=2\pi}sin^3\varphi d\varphi + C^4\int_{\varphi=0}^{\varphi=2\pi}sin^4\varphi d\varphi - C^5\int_{\varphi=0}^{\varphi=2\pi}sin^5\varphi d\varphi + C^6\int_{\varphi=0}^{\varphi=2\pi}sin^6\varphi d\varphi - \;\ldots\ldots\ldots\ldots\;\infty\right]$$

$$\textbf{using property of limit} \;\; \int_{\mathbf{0}}^{\mathbf{2a}}\boldsymbol{f(x)\,dx = 0 \; if \; f(2a-x) = -f(x)}$$

$$\mathbf{= 2}\int_{\mathbf{0}}^{\boldsymbol{a}}\boldsymbol{f(x)\,dx \;\; if \; f(2a-x) = f(x)}$$

$$\therefore \int_{\varphi=0}^{\varphi=2\pi}sin\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi}sin^3\varphi d\varphi = \int_{\varphi=0}^{\varphi=2\pi}sin^5\varphi d\varphi = \;\ldots\ldots\ldots.. = 0 \;\; \&$$

$$\int_0^{2\pi}sin^{2n}\varphi d\varphi = 2\int_0^{\pi}sin^{2n}\varphi d\varphi = 4\int_0^{\frac{\pi}{2}}sin^{2n}\varphi d\varphi \qquad \forall \; n \in N$$

Now, we have

$$P = \frac{htan\alpha}{sin^2\theta}\left[2\pi + 4C^2\int_0^{\frac{\pi}{2}} sin^2\varphi d\varphi + 4C^4\int_0^{\frac{\pi}{2}} sin^4\varphi d\varphi + 4C^6\int_0^{\frac{\pi}{2}} sin^6\varphi d\varphi + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2htan\alpha}{sin^2\theta}\left[\pi + 2C^2\int_0^{\frac{\pi}{2}} sin^2\varphi d\varphi + 2C^4\int_0^{\frac{\pi}{2}} sin^4\varphi d\varphi + 2C^6\int_0^{\frac{\pi}{2}} sin^6\varphi d\varphi + \dots\dots\dots\dots \infty\right]$$

Now, using $\boldsymbol{\beta}$ & $\gamma$**-functions**

$$\int_0^{\frac{\pi}{2}} \boldsymbol{sin^m x cos^n x\, dx} = \frac{\boldsymbol{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}}{\boldsymbol{2\Gamma\left(\frac{m+n+2}{2}\right)}} \quad \& \ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Hence, we have

$$P = \frac{2htan\alpha}{sin^2\theta}\left[\pi + 2\left(\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2+0+2}{2}\right)}\right)C^2 + 2\left(\frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)}\right)C^4 + 2\left(\frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{6+0+2}{2}\right)}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2htan\alpha}{sin^2\theta}\left[\pi + 2\left(\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)}\right)C^2 + 2\left(\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)}\right)C^4 + 2\left(\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2htan\alpha}{sin^2\theta}\left[\pi + 2\left(\frac{\frac{1}{2}\sqrt{\pi}\times\sqrt{\pi}}{2\times 1!}\right)C^2 + 2\left(\frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\sqrt{\pi}}{2\times 2!}\right)C^4 + 2\left(\frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\sqrt{\pi}}{2\times 3!}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2htan\alpha}{sin^2\theta}\left[\pi + 2\times\frac{\pi}{2}\left(\frac{\frac{1}{2}}{1!}\right)C^2 + 2\times\frac{\pi}{2}\left(\frac{\frac{3}{2}\times\frac{1}{2}}{2!}\right)C^4 + 2\times\frac{\pi}{2}\left(\frac{\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}}{3!}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2htan\alpha}{sin^2\theta}\times\pi\left[1 + \left(\frac{1}{2\times 1!}\right)C^2 + \left(\frac{1\times 3}{2^2\times 2!}\right)C^4 + \left(\frac{1\times 3\times 5}{2^3\times 3!}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$= \frac{2\pi htan\alpha}{sin^2\theta}\left[1 + \left(\frac{1}{2\times 1!}\right)C^2 + \left(\frac{1\times 3}{2^2\times 2!}\right)C^4 + \left(\frac{1\times 3\times 5}{2^3\times 3!}\right)C^6 + \dots\dots\dots\dots \infty\right]$$

$$\Rightarrow \boldsymbol{P = \left(\frac{2\pi htan\alpha}{sin^2\theta}\right)F_P} \qquad \dots\dots\dots\dots\dots\dots\dots\dots.. \ (VII)$$

Above is the required expression for calculating the perimeter of elliptical section of oblique frustum.

Where, $F_P \rightarrow$ is called Factor of surface area or **HCR's $\boldsymbol{F_P}$ series**

$$F_P = 1 + \left(\frac{1}{2\times 1!}\right)C^2 + \left(\frac{1\times 3}{2^2\times 2!}\right)C^4 + \left(\frac{1\times 3\times 5}{2^3\times 3!}\right)C^6 + \dots\dots\dots\dots \infty$$

**In generalised form**

$$F_P = 1 + \sum_{n=1}^{\infty}\left[\left\{\frac{1 \times 3 \times 5 \times \dots\dots\dots.(2n-3) \times (2n-1)}{2^n \times n!}\right\} C^{2n}\right]$$

$$\Rightarrow \boldsymbol{F_P = 1 + \sum_{n=1}^{\infty}\left[\left\{\frac{1 \times 3 \times 5 \times \dots\dots\dots.(2n-3) \times (2n-1)}{2^n \times n!}\right\} C^{2n}\right]} \qquad \dots\dots\dots\dots.(VIII)$$

Above series can be simplified as follows

$$\boldsymbol{F_P = 1 + 0.5C^2 + 0.375C^4 + 0.3125C^6 + 0.2734375C^8 + 0.24609375C^{10} + 0.225585937C^{12} + 0.209472656C^{14} + \dots\dots\dots\dots\dots\dots.\infty}$$

## V. CONCLUSION

Thus, all the results can concluded from equations III, IV, V, VI, VII & VIII as follows

Let a right circular cone with apex angle $2\alpha$ be thoroughly cut by a plane inclined at an angle $\theta$ with axis OO' of the cone & lying at a normal distance h from the apex point O (as shown in the fig 4 below)

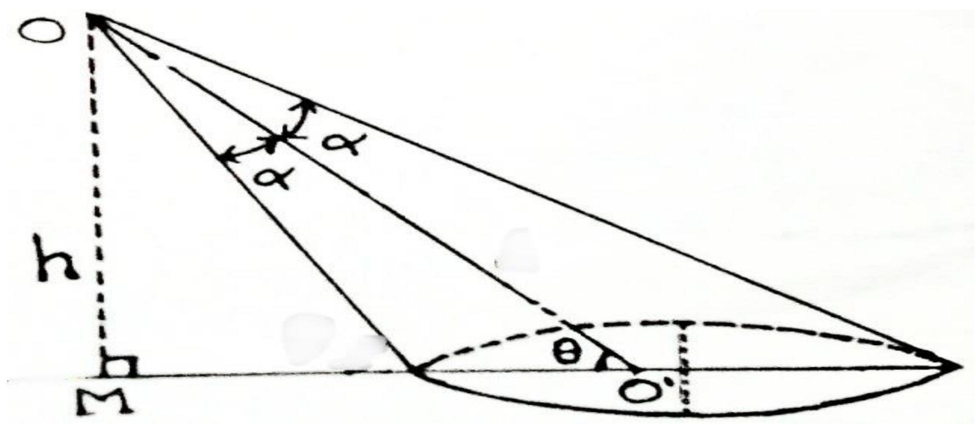


Fig. 4

Now, consider the oblique frustum with elliptical section & apex point 'O' (see above fig 4)

The following parameters can be determined as tabulated below

| | |
|---|---|
| **Volume (V) of frustum** | $$\boldsymbol{V = \frac{\pi h^3 sin^3\theta sin^2\alpha cos^4\alpha}{(sin^2\theta - sin^2\alpha)^3} - \left(\frac{2\pi h^3 tan^2\alpha}{3sin^3\theta}\right) F_V}$$ **where** $$\boldsymbol{F_V = 1 + \sum_{n=1}^{\infty}\left[(n+1)(2n+1)\left\{\frac{1 \times 3 \times 5 \times \dots\dots\dots.(2n-3) \times (2n-1)}{2^n \times n!}\right\} C^{2n}\right]}$$ $$\boldsymbol{= 1 + 3C^2 + 5.625C^4 + 8.75C^6 + 12.3046875C^8 + 16.2421875C^{10} + 20.52832031C^{12} + 25.13671875C^{14} + \dots\dots\dots\dots\dots\dots.\infty}$$ |
| **Surface Area (S) of frustum** | $$\boldsymbol{S = \left(\frac{\pi h^2 sec\alpha tan\alpha}{sin^2\theta}\right) F_S}$$ |

| | |
|---|---|
| | where $$F_S = 1 + \sum_{n=1}^{\infty}\left[(2n+1)\left\{\frac{1\times3\times5\times\dots\dots\dots(2n-3)\times(2n-1)}{2^n\times n!}\right\}C^{2n}\right]$$ $$= 1 + 1.5C^2 + 1.875C^4 + 2.1875C^6 + 2.4609375C^8 + 2.70703125C^{10} + 2.932617188C^{12} + 3.142089844C^{14} + \dots\dots\dots\dots\dots\dots\infty$$ |
| **Perimeter (P) of Elliptical section** | $$P = \left(\frac{2\pi h tan\alpha}{sin^2\theta}\right)F_P$$ where $$F_P = 1 + \sum_{n=1}^{\infty}\left[\left\{\frac{1\times3\times5\times\dots\dots\dots(2n-3)\times(2n-1)}{2^n\times n!}\right\}C^{2n}\right]$$ $$= 1 + 0.5C^2 + 0.375C^4 + 0.3125C^6 + 0.2734375C^8 + 0.24609375C^{10} + 0.225585937C^{12} + 0.209472656C^{14} + \dots\dots\dots\dots\dots\dots\infty$$ |
| **Major Axis (2a)** | $$2a = \frac{hsin2\alpha}{(sin^2\theta - sin^2\alpha)}$$ |
| **Minor Axis (2b)** | $$2b = \frac{2hsin\alpha}{\sqrt{(sin^2\theta - sin^2\alpha)}}$$ |
| **Eccentricity (e)** | $$e = \frac{cos\theta}{cos\alpha}$$ |
| **Conditions for known quantities $h, \alpha$ & $\theta$** | $$0 < \alpha < \theta \le \frac{\pi}{2} \ \& \ C = \left(\frac{tan\alpha}{tan\theta}\right) < 1$$ where $\alpha \rightarrow$ ***semi apex angle of right circular cone*** $\theta \rightarrow$ ***angle between cutting plane*** & *axis of cone* $h \rightarrow$ ***normal distance of cutting plane from the apex point of cone*** |

## VI. IMPORTANT DEDUCTIONS

1. All the series converge to a finite value for the given values of $\alpha$ & $\theta$. Although, the order of convergence rates of $\boldsymbol{F_V}$ , $\boldsymbol{F_S}$ & $\boldsymbol{F_P}$ is as follows $F_V \le F_S \le F_P$ & $F_V = F_S = F_P = 1\ for\ \theta = 90^o$

2. It is obvious that the rate of convergence of $F_V$ & $F_S$ depends on the values of $\alpha$ & $\theta$

   a. If $\theta - \alpha > 5^0 \Rightarrow F_V$ & $F_S$ converge to finite values with less number of terms i.e. terms of higher power can be neglected with insignificant errors in the results

   b. If $\theta - \alpha \le 5^0 \Rightarrow F_V$ & $F_S$ converge to finite values with more number of terms i.e. neglecting the terms of higher power may cause some error. In this case, the values of $F_V$ & $F_S$ must be determined by taking more number of terms until higher power terms become significantly negligible.

3. If a right circular cone with apex angle $2\alpha$ be thoroughly cut by a plane normal to the axis of the cone & lying at a normal distance h from the apex point then by setting $\theta = 90^o$ in the above results, we have the following results for the frustum generated

**a. Volume of frustum**

$$V=\frac{\pi h^3 sin^3 90^o sin^2\alpha cos^4\alpha}{(sin^2 90^o - sin^2\alpha)^3}-\left(\frac{2\pi h^3 tan^2\alpha}{3sin^3 90^o}\right)F_V$$

$$=\frac{\pi h^3 sin^2\alpha cos^4\alpha}{(1-sin^2\alpha)^3}-\left(\frac{2\pi h^3 tan^2\alpha}{3}\right)\times 1 \quad (since, F_V = 1\ for\ \theta = 90^o)$$

$$=\frac{\pi h^3 sin^2\alpha cos^4\alpha}{cos^6\alpha}-\frac{2\pi h^3 tan^2\alpha}{3}=\pi h^3 tan^2\alpha-\frac{2\pi h^3 tan^2\alpha}{3}$$

$$=\frac{1}{3}\pi h^3 tan^2\alpha=\frac{1}{3}\pi(htan\alpha)^2\times h=\frac{1}{3}\pi(radius)^2\times(normal\ height)$$

$$= volume\ of\ a\ right\ circular\ cone\ with\ apex\ angle\ 2\alpha\ \&\ normal\ height\ h$$

**b. Surface area of frustum**

$$S=\left(\frac{\pi h^2 sec\alpha tan\alpha}{sin^2 90^o}\right)F_S$$

$$=\left(\frac{\pi h^2 sec\alpha tan\alpha}{1}\right)\times 1 \qquad (since, F_S = 1\ for\ \theta = 90^o)$$

$$=\pi h^2 sec\alpha tan\alpha=\pi(htan\alpha)(hsec\alpha)=\pi\times(radius)\times(slant\ height)$$

$$= surface\ area\ of\ right\ circular\ cone\ with\ apex\ angle\ 2\alpha\ \&\ normal\ height\ h$$

**c. Perimeter of section generated**

$$P=\left(\frac{2\pi htan\alpha}{sin^2 90^o}\right)F_P$$

$$=\left(\frac{2\pi htan\alpha}{1}\right)\times 1=2\pi(htan\alpha)=2\pi\times(radius) \qquad (since, F_P = 1\ for\ \theta = 90^o)$$

$$= periphery\ of\ base\ of\ right\ circular\ cone\ with\ apex\ angle\ 2\alpha\ \&\ normal\ height\ h$$

**d. Major axis of generated section**

$$2a=\frac{hsin2\alpha}{(sin^2 90^o - sin^2\alpha)}$$

$$=\frac{hsin2\alpha}{(1-sin^2\alpha)}=\frac{2hsin\alpha cos\alpha}{cos^2\alpha}=2(htan\alpha)=2\times(radius)$$

$$= diameter\ of\ base\ of\ right\ circular\ cone\ with\ apex\ angle\ 2\alpha\ \&\ normal\ height\ h$$

**e. Minor axis of generated section**

$$2b=\frac{2hsin\alpha}{\sqrt{(sin^2 90^o - sin^2\alpha)}}$$

$$=\frac{2hsin\alpha}{\sqrt{(1-sin^2\alpha)}}=\frac{2hsin\alpha}{cos\alpha}=2(htan\alpha)=2\times(radius)$$

$$= diameter\ of\ base\ of\ right\ circular\ cone\ with\ apex\ angle\ 2\alpha\ \&\ normal\ height\ h$$

**f. Eccentricity of generated section**

$$e=\frac{cos90^o}{cos\alpha}=0=eccentricity\ of\ a\ circle$$

It is clear from the above results that when a right circular cone is thoroughly cut with a plane normal to the axis of cone generates a frustum which is itself a right cone with circular section. Hence, all the mathematical results derived by the author are verified.

Thus, the above results are very useful for analysis of oblique frustum in determining the volume, surface area & perimeter of elliptical section. All the derived results can be easily verified by the experimental results. These results can be used in practical applications, case studies & other academic purposes.

## VII. ILLUSTRATIVE EXAMPLE

Consider a right circular cone, with apex angle $2\alpha = 60^o$, is thoroughly cut by a smooth plane inclined at an angle $\theta = 70^o$ with the axis & lying at a normal distance $h = 20cm.$ from the apex point.

In this case, $\theta - \alpha = 70^o - 30^o = 40^o >> 5^o$ hence the rate of convergence of $F_V, F_S$ & $F_P$ series is much higher i.e. terms of higher power can be neglected with no significant error in the results

Let's first calculate the values of $F_V, F_S$ & $F_P\ for\ \alpha = 30^o$ & $\theta = 70^o$ as follows

$$F_V = 1 + 3C^2 + 5.625C^4 + 8.75C^6 + 12.3046875C^8 + 16.2421875C^{10} + 20.52832031C^{12} + 25.13671875C^{14} + \dots\dots\dots\dots\dots\dots.\infty$$

$$On\ setting\ the\ value\ of\ constant, C = \left(\frac{tan30^o}{tan70^o}\right), we\ have$$

$$F_V = 1 + 3\left(\frac{tan30^o}{tan70^o}\right)^2 + 5.625\left(\frac{tan30^o}{tan70^o}\right)^4 + 8.75\left(\frac{tan30^o}{tan70^o}\right)^6 + 12.3046875\left(\frac{tan30^o}{tan70^o}\right)^8 + 16.2421875\left(\frac{tan30^o}{tan70^o}\right)^{10} + 20.52832031\left(\frac{tan30^o}{tan70^o}\right)^{12} + 25.13671875\left(\frac{tan30^o}{tan70^o}\right)^{14} + \dots\dots\dots\dots\dots\dots.\infty$$

$$= 1 + 0.132474331 + 0.010968405 + 0.00075342440828 + 0.00004678563359 + 0.000002727074033 + 0.0000001522005794 + 0.000000008229661223 + \dots\dots\dots\dots.\infty$$

$$\Rightarrow \boldsymbol{F_V \approx 1.144245834}$$

$$similarly, F_S = 1 + 1.5\left(\frac{tan30^o}{tan70^o}\right)^2 + 1.875\left(\frac{tan30^o}{tan70^o}\right)^4 + 2.1875\left(\frac{tan30^o}{tan70^o}\right)^6 + 2.4609375\left(\frac{tan30^o}{tan70^o}\right)^8 + 2.70703125\left(\frac{tan30^o}{tan70^o}\right)^{10} + 2.932617188\left(\frac{tan30^o}{tan70^o}\right)^{12} + 3.142089844\left(\frac{tan30^o}{tan70^o}\right)^{14} + \dots\dots\dots\dots\dots\dots.\infty$$

$$= 1 + 0.066237165 + 0.003656135102 + 0.0001883560207 + 0.000009357126717 + 0.0000004545123389 + 0.00000002174293992 + 0.000000001028707653 + \dots\dots..\dots\dots\dots.\infty$$

$$\Rightarrow \boldsymbol{F_S \approx 1.070091491}$$

$$similarly, F_P = 1 + 0.5\left(\frac{tan30^o}{tan70^o}\right)^2 + 0.375\left(\frac{tan30^o}{tan70^o}\right)^4 + 0.3125\left(\frac{tan30^o}{tan70^o}\right)^6 + 0.2734375\left(\frac{tan30^o}{tan70^o}\right)^8 + 0.24609375\left(\frac{tan30^o}{tan70^o}\right)^{10} + 0.225585937\left(\frac{tan30^o}{tan70^o}\right)^{12} + 0.209472656\left(\frac{tan30^o}{tan70^o}\right)^{14} + \dots\dots\dots\dots\dots\dots.\infty$$

$$= 1 + 0.022079055 + 0.0007312270203 + 0.00002690800296 + 0.000001046928806 + 0.00000004131930353 + 0.000000001672533836 + 0.00000000006858051011 + \dots\dots\dots\dots\dots..\infty$$

$$\Rightarrow \boldsymbol{F_P \approx 1.02283828}$$

Now, on setting the corresponding values of $\alpha, \theta, h, F_V, F_S$ & $F_P$ we get the following results

**a. Volume of frustum**

$$V = \frac{\pi h^3 sin^3\theta sin^2\alpha cos^4\alpha}{(sin^2\theta - sin^2\alpha)^3} - \left(\frac{2\pi h^3 tan^2\alpha}{3sin^3\theta}\right)F_V$$

$$= \frac{\pi(20)^3 sin^3 70^o sin^2 30^o cos^4 30^o}{(sin^2 70^o - sin^2 30^o)^3} - \left(\frac{2\pi(20)^3 tan^2 30^o}{3sin^3 70^o}\right) \times 1.144245834$$

$$= \mathbf{3859.458211\ cm^3}$$

**b. Surface area of frustum**

$$S = \left(\frac{\pi h^2 sec\alpha tan\alpha}{sin^2\theta}\right)F_S$$

$$= \left(\frac{\pi(20)^2 sec30^o tan30^o}{sin^2 70^o}\right) \times 1.070091491 = \mathbf{1015.238041\ cm^2}$$

**c. Perimeter of section generated**

$$P = \left(\frac{2\pi h tan\alpha}{sin^2\theta}\right)F_P$$

$$= \left(\frac{2\pi(20)tan30^o}{sin^2 70^o}\right) \times 1.02283828 = \mathbf{82.53924536\ cm}$$

**d. Major axis of generated section**

$$2a = \frac{hsin2\alpha}{(sin^2 90^o - sin^2\alpha)}$$

$$= \frac{(20)sin(2 \times 30^o)}{(sin^2 70^o - sin^2 30^o)} = \mathbf{27.36161147\ cm}$$

**e. Minor axis of generated section**

$$2b = \frac{2hsin\alpha}{\sqrt{(sin^2\theta - sin^2\alpha)}}$$

$$= \frac{2(20)sin30^o}{\sqrt{(sin^2 70^o - sin^2 30^o)}} = \mathbf{25.13740937\ cm}$$

**f. Eccentricity of generated section**

$$e = \frac{cos70^o}{cos30^o} = \mathbf{0.394930843}$$

Thus, all the mathematical results obtained above can be verified by the experimental results. The symbols & names used above are arbitrary given by the author Mr Harish Chandra Rajpoot (B Tech, Mech. Engg.).

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## REFERENCES

The work of the author is original. It is an outcome of deep studies & experiments carried out by the author on the analysis of oblique frustum of right circular cone.

[1] Some results are directly taken from “Advanced Geometry”, a book of research articles, by the author published with Notion Press, India (www.notionpress.com)

[2] Previous paper entitled “HCR’s Rank Formula” was subjected to the peer review by Dr K. Srinivasa Rao, Senior professor, Distinguished Professor (DST-Ramanujan Prof.), director of Ramanujan Academy of Maths Talent, Luz Church Road, Chennai, India.